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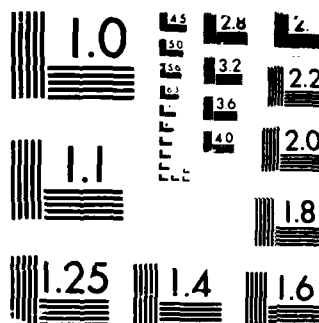
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Optimal Design of Fibered Structures

Final Technical Report

Gilbert Strang

February 1988

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## Optimal Design of Fibered Structures

### ABSTRACT

The problem of shape optimization is to minimize the area that must be filled by expensive material in order to satisfy design constraints. Those constraints require the structure to withstand a given external load (or family of loads) without exceeding a permissible deformation, or a permissible compliance, or the yield limit of a plastic material.

The design problem is mathematically subtle, because the underlying optimization problem is not convex. Structural equilibrium is governed by a partial differential equation of the form  $\text{div}(c(x)\text{grad } u) = f$ , but in contrast to the analysis problem (which solves for  $u$ ), the design problem is to choose  $c$ . The control variable is the coefficient, and frequently a conventional solution does not exist. The minimum weight design is approached by coefficients  $c$  which jump more and more frequently between alternative states. The design develops a complicated microstructure, and the challenge is to see within that structure a simple and computable pattern.

The limit is a composite material, in which the original materials have well-defined densities and orientations. The composite is achieved by homogenization of the original materials. Mathematically this is expressed by a relaxation, or convexification, of the original minimization problem. The project has seen a successful integration of these fundamental theories (previously developed along separate lines), and the explicit computation of optimal designs in a series of significant engineering problems.

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Problem and Results: Technical Report

Contract DAAG29-83-K-0025

Optimal Design of Fibered Structures

Professor Gilbert Strang  
Department of Mathematics  
Massachusetts Institute of Technology

The classical variational problems involve the minimization of convex energies subject to boundary conditions. They lead to elliptic differential equations and the solutions are smooth functions away from singularities in the data or the domain. A typical equation is  $\text{div} (c(x) \text{grad } u) = f$ , with a coefficient  $c$  that reflects the material properties.

In contrast, the problem of optimal design is to choose that coefficient  $c$ . The cost function may still be quadratic in  $u$ , or in plasticity problems it may involve the  $L^\infty$  norm of  $c \text{ grad } u$ , but the choice between different materials or different shapes is determined by the unknown function  $c$ : it is a problem in which the coefficient is the control variable. In such problems it is common to find that a conventional solution does not exist. The minimum cost is approached by a sequence  $c_n$  that is more and more oscillatory. That corresponds to a series of designs in which materials are combined with an increasingly complicated microstructure.

Mathematically the underlying difficulty is that the variational problem is not convex. In optimal design the cost may be zero in the absence of material and jump to unity whenever the stress is nonzero and material is required. The minimizing sequence of coefficients makes that jump more and more frequently, and there is no convergence in norm to a limit  $c_\infty(x)$ . However there is weak convergence. The quantity of material in each region does approach a limit, and the orientation can be defined. The limit can be described as a

composite material, achieved by homogenization of the original materials.

The mathematical theory is expressed by a relaxation of the original nonconvex problem. The problem is replaced by a well-posed (lower semicontinuous) problem which has the same minimum cost, but that minimum value is actually achieved (by a composite material). The solution of the relaxed problem is the weak limit of minimizing sequences in the original problem. Our joint work with Robert Kohn, referenced in the bibliography below, carried out an explicit relaxation and solution of problems in optimal design. It is comparatively straightforward in the case of scalar problems, but it becomes more complicated when the unknown  $u$  is a vector function. Elasticity and continuum mechanics produce the vector case, and much of our research was concentrated on the relaxation of functionals  $\int F(\text{grad } u) dx$ , in which  $\text{grad } u$  is the Jacobian matrix and quasiconvexity is the key property to achieve for  $F$ .

The long paper [1] summarizes our work on the interaction of homogenization, relaxation, and the mathematical theory of composite materials. In this report I will describe one specific problem solved separately in [4]. It was a central problem in Tartar's development of the method of compensated compactness, and it is an optimization problem for the effective properties of composites: How should conducting material be placed in a unit square so as to carry prescribed currents in the north-south and east-west directions, with a minimum area of material? We formulated this question as a nonconvex variational problem:

Minimize the area of the subset  $S \subset [0,1]^2$  subject to

$\iint |\nabla u|^2 dx dy = Q$ ,  $\iint |\nabla v|^2 dx dy = R$ ,  $\nabla u = \nabla v = 0$  outside  $S$ ,  
with boundary conditions  $u = 0$  on the left side,  $u = 1$  on the right,  $v = 0$  on the bottom, and  $v = 1$  on the top. The

stream functions are  $u$  and  $v$  (a vector unknown), the effective resistances of the square are  $Q$  and  $R$ , and the problem is to achieve those resistances with a minimum of material. The area  $(Q+R-2)/(QR-1)$  can be achieved by a suitable microstructure and the problem is to prove that this area is minimal.

The nonconvexity enters in the expression  $\iint l_S \, dx dy$  for the area; the characteristic function equals one in  $S$  and zero in the complement. The difficulty is to compute the "quasiconvexification" of the functional  $F(vu, vv) =$

$l_S + \lambda |vu|^2 + \mu |vv|^2$ , in which the Lagrange multipliers  $\lambda$  and  $\mu$  enter from the constraints. Kohn and I determined the largest function  $PF$  below  $F$  that is polyconvex (convex in  $vu, vv$ , and  $\det J$ ). We also established that this function is convex on lines connecting Jacobian matrices  $J = [vu \, vv]$  that differ by rank one. Therefore  $PF$  is the correct quasiconvexification of  $F$ , and the relaxed problem requires the minimization of  $\iint PF(vu, vv) \, dx dy$ . That calculation led back to  $(Q+R-2)/(QR-1)$  as a lower bound on the area, and therefore the minimum.

## Publications (Gilbert Strang)

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### IV. Textbook on applied and engineering mathematics

12. Introduction to Applied Mathematics, Wellesley-Cambridge Press (1986).

Scientific Collaborators

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